this is where the need for stabilizing enters. Specifically, one shows that $\phi \times p$ is equivalent to $p \times \phi'$, where ϕ' comes from the differential of ϕ, which is linear over the first Z-factor.

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SPHERICAL VARIETIES AN INTRODUCTION

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When one studies complex algebraic homogeneous spaces it is natural to begin with the ones which are complete (i.e. compact) varieties. They are the "generalized flag manifolds" . Their occurence in many problems of representation theory, algebraic geometry, ... make them an important class of algebraic varieties. In order to study a noncompact homogeneous space G/H, it is equally natural to compactify it, i.e. to embed it (in a G-equivariant way) as a dense open set of a complete G-variety . A general theory of embeddings of homogeneous spaces has been developed by Luna and Vust [LV]. It works especially well in the so-called spherical case: G is reductive connected and a Borel subgroup of G has a dense orbit in G/H. (This class includes complete homogeneous spaces as well as algebraic tori and symmetric spaces). A nice feature of a spherical homogeneous space is that any embedding of it (called a spherical variety) contains only finitely many G-orbits, and these are themselves spherical. So we can hope to describe these embeddings by combinatorial invariants, and to study their geometry. I intend to present here some results and questions on the geometry (see [LV], [BLV], [BP], [Lun] for a classification of embeddings).

In § 1 I give a local description of a spherical variety, and draw some consequences about its singularities. I also introduce the rank of a spherical homogeneous space G/H; it is the minimal codimension of a K-orbit, where K is a maximal compact subgroup of G. It measures how far G/H is from being complete.

In § 2 I study the topology of a spherical variety, using a cellular decomposition (which generalizes the Bruhat decomposition of flag manifolds). The example of "complete conics" is studied in some detail in 2.4, because of its historical significance.

In § 3 I describe some connections between spherical varieties and moment mappings. They lead to a solution of the following problem: Given a (smooth projective) spherical variety X, of dimension n, describe its second cohomology group $H^2(X, \mathbf{C})$, and compute products of n arbitrary elements of this group. There are interesting applications to classical enumerative problems (see 3.6 and [**DP1**], [**B4**]). As an example, I describe the cohomology algebra of complete conics (3.7).

One of my aims is to call attention to the numerous unsolved problems concerning spherical varieties. The most important one seems to be: to generalize the "Schubert calculus" (which gives a nice picture of the cohomology of a complete homogeneous space). I refer to [**DP3**], [**DGMP**], [**BDP**] for some partial solutions. The reader will find other problems at the end of each section.

Proofs of theorems will be omitted, or only sketched. The result in 3.3 is new, but the proof I know is too long to be published here.

§ 1. Local properties of spherical varieties

1.1. Definitions.

Let G be a connected reductive algebraic group over $\mathbf{C}$. An algebraic subgroup H of G is said to be *spherical* if some Borel subgroup B of G has a dense orbit in the homogeneous space G/H (which is also said to be spherical). We can arrange that the product BH is open in G, and all Borels having this property form a single H-conjugacy class. We call them "opposite Borel subgroups."

Choose an opposite B, and denote by P the set of all $s \in G$ such that $sBH = BH$. Then P is a parabolic subgroup of G which contains B. It will play an important role in the sequel.

1.2. Examples.

(i) G is an algebraic *torus*, and H the trivial group (here any Borel subgroup of G is equal to $G\ldots$).

(ii) G is any connected reductive group, and H is a *parabolic subgroup*. Denote by H^u its unipotent radical. Then a Borel subgroup B is opposite if and only if $B \cap H^u = \{1\}$. Then the parabolic subgroup P is opposite to H in the usual meaning (i.e. $P \cap H$ is maximal reductive in both P and H).

(iii) G is connected reductive. Let σ be an involution of G, and H the subgroup of its fixed points. Then G/H is a *symmetric space*. A Borel subgroup B is opposite exactly when the intersection $B \cap \sigma(B)$ has minimal dimension. P is a minimal σ-split parabolic subgroup, i.e. P and $\sigma(P)$ are opposite, and P is minimal with this property ([**BLV**] 4.1).

There is a classification of *reductive* spherical subgroups H of a reductive group G [**Mik**], [**B2**]. Such "spherical pairs" (G, H) are complexifications of pairs (K, L) where K and L are compact Lie groups and the K-module $L^2(K/L)$ is multiplicity-free [**GS3**].

1.3. The problem of local retractions.

Let G act on an algebraic variety X over $\mathbf{C}$, such that the map $(g, x) \to g.x$ is algebraic (X will be called a *G-variety*). Let $Y = G.x$ be the G-orbit of a given $x \in X$. In general it is difficult to describe the action of G near Y, because (unlike for actions of compact groups) there is no equivariant retraction of a neighbourhood of Y onto Y. Here is an easy example: Let $G = SL_2(\mathbf{C})$ .The linear action of G on the space of quadratic forms in two variables induces an action of G on the complex projective plane X. The conic Y of degenerate quadratic forms, and its complement $X - Y$ are G-orbits. The isotropy group of a point of Y is a Borel subgroup of G, while the isotropy group of a point of $X - Y$ is the normalizer of a torus and is not contained in any Borel. Hence there is no equivariant retraction of a neighbourhood of Y onto Y. However, there is almost a retraction: Let $y \in Y$, let l be the tangent line to Y at y, and let B be the isotropy group of y. Then there is a B-equivariant retraction $r : X - l \to Y - \{y\}$: It associates to $x \in X - l$ the unique $r(x) \in Y - \{y\}$ such that x, y, $r(x)$ are on the same line.

1.4. Action of an algebraic group near a spherical orbit.

Let G, X, Y as in the beginning of 1.3. In view of the preceding example it seems reasonable to try to describe the action of a parabolic subgroup of G on an open set of X which meets Y. When the orbit Y is spherical, we have the following result [**B4**; 1.1].

Theorem. *Let X be a normal G-variety with spherical G-orbit Y. Choose $y \in Y$ and denote by H its isotropy subgroup. Let B, P be as in 1.1, and denote by P^u the unipotent radical of P. Then there exists a Levi subgroup L of P, an open P-stable subset X' of X and a subvariety Z of X' such that*

(i) *Z is affine, L-stable and contains y. Moreover, $Z \cap Y$ is equal to $L.y$, and is closed in Z.*

(ii) *The derived subgroup (L, L) of L fixes y.*

(iii) *The map $(s, z) \in P^u \times Z \to s.z \in X'$ is an isomorphism.*

Furthermore, there exists an affine subvariety Z' of Z, containing Y and stable by the isotropy subgroup L_y, such that $Z' \cap Y = \{y\}$ and such that the map $L \times_{L_y} Z' \to Z$ is an isomorphism.

As a corollary, we obtain the existence of a P-equivariant "retraction" of X' onto $X' \cap Y = P.y$. Moreover, the map $(s, z) \in P^u \times L.y \to s.z \in P.y = B.y$ is an isomorphism.

1.5. Local structure of spherical varieties.

Let us say that the G-variety X is *spherical* if X is normal and some Borel subgroup B of G has a dense orbit. (For $G = T$ a torus this simply means that T has a dense orbit in X. Such varieties, called *toric varieties*, have been thoroughly studied. See [**Oda**] and references therein). Theorem 1.4 can be refined for spherical varieties.

Theorem (see [**B4**], 1.1 and 1.2). *Let X be a spherical variety.*

(i) *X contains only finitely many G-orbits, all of which are spherical. In fact, X contains only finitely many B-orbits [**B1**], [**Vin**]. Let Y, y, H, B and P be as in 1.4. Denote by $X_{Y,B}$ the set of all $x \in X$ such that Y is contained in the closure of the orbit $B.x$.*

(ii) *We can take $X' = X_{Y,B}$ in Theorem 1.4. Moreover, Z (resp. Z') is a spherical L-(resp. L_y-)variety.*

This result means that along a G-orbit a spherical variety looks like an affine spherical variety with a fixed point . As a consequence one can prove that singularities of spherical varieties are "not too bad". Recall that a normal variety V is said to have *rational singularities* if there exists a resolution of singularities $\pi : V' \to V$ such that $R^i\pi_*\mathcal{O}_{V'}$ vanishes for every nonzero i (then this property holds for any resolution).

Corollary ([**B4**] 1.2, [**Pop**] Theorem 10). *The singularities of spherical varieties are rational.*

1.6. Rank of a spherical variety.

Let G, H, B, P, L be as in 1.4. Denote by C the identity component of the center of L. Then the open B-orbit in G/H is isomorphic to $P^u \times (L/L \cap H) = P^u \times (C/C \cap H)$ because L is equal to the product $C.(L, L)$, and (L, L) is contained in $L \cap H$ (Theorem 1.4 (ii)). Define the *rank* of G/H to be the dimension of $C/C \cap H$.

Let us compute the rank for the examples of Part 1.2.

(i) When G is a torus and H the trivial group, then the rank is the dimension of G.

(ii) If H is a parabolic subgroup of G, then the rank of G/H is zero (and this characterizes parabolic subgroups among spherical subgroups).

(iii) If H is the group of fixed points of an involution σ of G, then the rank of G/H has the usual meaning for a symmetric space: it is the dimension of a maximal σ -split torus of G (i.e. a torus A such that $\sigma(a) = a^{-1}$ for every $a \in A$, and maximal with this property).

Define the rank of a spherical variety to be the rank of its dense G-orbit (then one can show that the ranks of the other G-orbits are strictly smaller). By (ii), spherical varieties of rank zero are just complete homogeneous spaces. It can be shown that complete spherical varieties of rank one contain only two or three orbits. By blowing- up the closed orbits one obtains a completion of a homogeneous space by one or two homogeneous divisors. Such objects have been classified by D. N. Ahiezer [**Ahi**].

Let us conclude this section with two open questions on "spherical singularities".

(i) Let X be an affine spherical variety with a fixed point x. Does there exist a $\mathbf{C}^*$ -action on X which commutes with the G-action and has the point $\{x\}$ as unique closed orbit?

(ii) What are the singularities of spherical varieties of rank two ? (see [**B4**] 1.3 for the case of rank one). When G is a torus, one only gets quotients of $\mathbf{C}^2$ by finite cyclic groups ([**Oda**] 1.24).

§ 2. Topology of spherical varieties

2.1. The Bialynicki-Birula decomposition.

Let T be a complex torus acting on a variety X with only finitely many fixed points $x_1, \dots, x_r$. Then one can choose a one-parameter subgroup $\lambda : \mathbf{C}^* \to T$ such that the fixed points of T and $\mathbf{C}^*$ (acting via λ) are the same; such a λ is said to be "in general position". For $1 \leq i \leq r$ denote by $X(\lambda, x_i)$ the set of all $x \in X$ such that $\lambda(t).x \to x_i$ as $t \to 0$, and call it the *cell* of x_i. If X is projective and smooth, then ([**BB1**], [**BB2**]) each $X(\lambda, x_i)$ is isomorphic to some $\mathbf{C}^{n_i}$ for a non-negative integer n_i. So X has a paving by affine spaces. Moreover, $x_1, \dots, x_r$ can be ordered so that the closure of each $X(\lambda, x_i)$ is contained in the union of the $X(\lambda, x_j)$'s for $j \geq i$. Hence the cohomology algebra $H^*(X, \mathbf{Z})$ is a free group on the classes of closures $\overline{X(\lambda, x_i)}$. The Poincaré polynomial of X is $\sum_{i=1}^{r} t^{2n_i}$. If $A^*(X)$ denotes the Chow ring of X (i.e. the group of algebraic cycles on X modulo rational equivalence, the product of two transversal cycles being their intersection), then the natural map $A^*(X) \to H^*(X, \mathbf{Z})$ is an isomorphism.

All these results hold for a projective smooth spherical variety. Indeed, such a variety contains only finitely many G- orbits. Let T be a maximal torus of G. Then an easy lemma ([**DS**] 2.2) shows that T fixes only finitely many points in every homogeneous space of G. So T fixes only finitely many points of X. Proceeding as before we obtain a cellular decomposition of a projective smooth spherical variety, called the *Bialynicki-Birula decomposition*. For a complete homogeneous manifold it coincides with the Bruhat decomposition.

2.2. The Bialynicki-Birula decomposition and B-orbits.

Every spherical variety is a finite union of orbits of a given Borel subgroup of G (Theorem 1.5 (i)). As orbits of a connected solvable algebraic group, they are isomorphic to products of copies of $\mathbf{C}$ and $\mathbf{C}^*$. There is a second coarser decomposition of X into G-orbits, and there is also the Bialynicki-Birula decomposition. A connection between these three decompositions was discovered by DeConcini and Springer **[DS]** for certain compactifications of symmetric spaces , and their results were generalized in **[BL]** using different techniques. In order to state the result we need some more notation.

Choose a Borel subgroup B of G, and a maximal torus T of B. If $\lambda : \mathbf{C}^* \to T$ is a one-parameter subgroup, denote by $P(\lambda)$ the set of all $g \in G$ such that $\lambda(t)g\lambda(t)^{-1}$ has a limit in G as $t \to 0$. It is a parabolic subgroup of G. For a well-chosen λ (inside the "positive Weyl chamber" of T), the group $P(\lambda)$ is equal to B. It follows easily that for every G-variety and every fixed point x of T in X, the cell $X(\lambda, x)$ is B-stable.

We need the following

DEFINITION: A spherical G-variety X with open G-orbit X_G^0 is *toroidal* if the closure of every B-stable divisor in X_G^0 contains no G-orbit.

For example, toric varieties are always toroidal (!). The "complete symmetric varieties" studied by DeConcini, Procesi ... **[DP1]**, **[DP2]**, **[DS]** can be identified with the toroidal G-varieties whose open G-orbit is a symmetric space.

Theorem ([**BL**] 2.3). *Let X be a toroidal complete G-variety, and let B, T be as before. Then the intersection of any cell $X(\lambda, x)$ with any G-orbit is empty or is a single B-orbit.*

Note that X does not need to be smooth (but it needs to be toroidal; see **[BL]** 2.3). This result shows that B-orbits in X can be described in terms of G-orbits and fixed points of T. Unfortunately, it is in general difficult to compute them (see **[DS]** for complete symmetric varieties).

2.3. A description of the second cohomology group.

We will describe $H^2(X, \mathbf{Z})$ for a smooth complete spherical variety X, in a way which is independant of the Bianynicki- Birula decomposition, and more suited for applications.

Theorem. *Let X be a smooth complete spherical G-variety, and let B be a Borel subgroup of G.*

(i) *The group $H^2(X, \mathbf{Z})$ is generated by the classes of irreducible B-stable divisors of X. The relations are the divisors of the rational functions on X which are eigenvectors of B.*

(ii) *If G has only one closed orbit Y in X, then $H^2(X, \mathbf{Z})$ is freely generated by the classes of irreducible B-stable divisors which do not contain Y.*

REMARK: More generally, if X is any spherical variety (say with only one closed G-orbit Y), then the divisor class group of X is still generated by the classes of irreducible B-stable divisors, with relations as in (i). The Picard group of X is generated by the classes of irreducible B-stable divisors which do not contain Y (**[B4]** §2).

2.4. An example: complete conics.

The set Q consisting of all smooth conics in the complex projective plane $\mathbf{P}^2$ is the homogeneous space $PGL(3)/PO(3)$ where $PGL(3)$ is the automorphism group of $\mathbf{P}^2$, and $PO(3)$ is the projective orthogonal group of a nonsingular quadratic form in three variables. Q is a symmetric space of rank two. Let V be the space of all quadratic forms in three variables, and V^* its dual. Then the projective spaces $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$ are natural compactifications of Q, equivariant with respect to $G = PGL(3)$: We embed Q in $\mathbf{P}(V)$ (resp. $\mathbf{P}(V^*)$) by associating to each smooth conic C its line of equations (resp. the equations of the dual conic C^*, consisting of all tangent lines to C).The closure X of the set of all pairs (C, C^*) in $\mathbf{P}(V) \times \mathbf{P}(V^*)$ is called the space of *complete conics.* It is a (five-dimensional) smooth variety (**[DP1]** Theorem 3.1). We will describe the G- orbits in X, and compute its Poincaré polynomial.

There are three distinct degenerations of a pair (C, C^*), hence three G-orbits in X, outside the open orbit Q:

(i) C degenerates into two distinct lines l and l', and C^* into the set of all lines containing the point $l \cap l'$.

(ii) C^* degenerates into the set of all lines containing one of two fixed points p and p', and C into the double line joining p and p'.

(iii) C degenerates into a double line l, and C^* into the set of all double lines through a given point p on l.

In particular G has only one closed orbit Y in X (the complete conics of type (iii)), isomorphic to the flag variety of $\mathbf{P}^2$.

A maximal torus T of G is the isotropy group of three ordered points x, y, z in $\mathbf{P}^2$, not lying on the same line. It is easy to see that:

(a) T fixes no point of Q.

(b) There are three fixed points of T in the G-orbit of complete conics of type (i) (resp. (ii)), obtained by taking two sides of the triangle xyz as l and l' (resp. two vertices of this triangle as p and p').

(c) There are six fixed points of T in the complete conics of type (iii), i.e. the six flags defined by the triangle xyz.

Hence T fixes 12 points of X: the rank of $H^*(X, \mathbf{Z})$ is 12.

Let B be a Borel subgroup of G. Then B is the isotropy group of a flag (p, l) in $\mathbf{P}^2$. A B-orbit in Q is described by the relative position of a flag and a conic. Hence there are two B-orbits of codimension one in Q: the set of conics containing p (resp. tangent to l). From Theorem 2.3 (ii) we deduce that their closures (classically denoted by μ and ν) form a basis of $H^2(X, \mathbf{Z})$. In particular, its rank is equal to 2. From the fact that $H^*(X, \mathbf{Z})$ is of rank 12, and Poincaré duality, we conclude that *the Poincaré polynomial of X is* $1 + 2t^2 + 3t^4 + 3t^6 + 2t^8 + t^{10}$. The algebra $H^*(X, \mathbf{Z})$ will be described by generators and relations in 3.7 below.

§ 3. Spherical varieties and the moment map

3.1. The moment map.

Let V be a representation space of a connected reductive complex group G. Choose a maximal compact subgroup K of G, and a K-invariant positive definite hermitian form on V; we denote by a dot the associated scalar product. One can then canonically define a K-invariant Kähler structure on the projective space $\mathbf{P}(V)$, which is, in particular, a symplectic variety (the symplectic form ω being the imaginary part of the Kähler form). Define a map μ from X to $\mathcal{K}^*$ (the dual of the Lie algebra $\mathcal{K}$ of K) by $\langle \mu(x), A\rangle = (\tilde{x}.A\tilde{x})(\tilde{x}.\tilde{x})^{-1}$ where $x \in \mathbf{P}(V)$; $\tilde{x}$ is a representant of x in V; $A \in \mathcal{K}$. It is easy to see that μ is K-equivariant and that its differential $d\mu$ verifies: $\langle d\mu_x(\xi), A\rangle = \omega_x(\xi, A_x)$ for every $x \in X$, $\xi \in T_x\mathbf{P}(V)$, $A \in \mathcal{K}$. This means that μ is a *moment map* for the symplectic action of K on $\mathbf{P}(V)$ ([GS1&2], [Kir], [Nes]).

If X is any closed smooth algebraic subvariety of $\mathbf{P}(V)$ which is G-stable, then X inherits a K-invariant Kähler structure, and the restriction of μ to X is still a moment map for the induced symplectic structure on X.

The image $\mu(X)$ has a nice convexity property. As $\mu(X)$ is a K-stable subset of $\mathcal{K}^*$, it is described by its intersection with a fundamental domain C of K acting on $\mathcal{K}^*$. We can choose C to be a Weyl chamber in the dual $\mathcal{T}^*$ of a Cartan algebra $\mathcal{T}$ of $\mathcal{K}$.

Theorem ([GS1&2], [Kir], [Nes]). *The intersection $\mu(X) \cap C$ is a convex polyhedron with rational vertices.*

(Recall that $\mathcal{T}^*$ contains the lattice $\mathcal{T}^*_{\mathbf{Z}}$ of weights of the maximal torus T of K with Lie algebra $\mathcal{T}$. Hence we can speak of integral and rational points of $\mathcal{T}^*$).

3.2. A characterization of spherical varieties.

Keep the same notation as above. Let $\pi : \mathcal{K}^* \to \mathcal{K}^*/K$ be the quotient by K. The restriction of π to C is an isomorphism. The composition $\overline{\mu} = \pi \circ \mu$ is a K-invariant map from X to a linear space.

Theorem ([B3] 5.1). *Let X be a G-stable closed smooth subvariety of $\mathbf{P}(V)$. Then the following conditions are equivalent:*

(i) *X is spherical*

(ii) *The fibers of $\overline{\mu}$ are K-orbits.*

If either (i) or (ii) is verified, then the rank of X is the dimension of $\overline{\mu}(X)$, i.e. of $\mu(X) \cap C$. It is also the minimal codimension of the K-orbits in X.

For a smooth projective spherical G-variety X, we have thus a realization of the quotient by K as a map to a convex polyhedron (because we can identify $\overline{\mu}(X)$ and $\mu(X) \cap C$). This raises a number of questions: is this map a quotient in the C^∞ category ? Is it possible to obtain topological information about X from the contractibility of X/K ?

3.3. An integral formula.

Consider as before a closed G-subvariety X of $\mathbf{P}(V)$. Let Γ be the intersection of $\mu(X)$ with the Weyl chamber $C \subset \mathcal{T}^*$. Denote by R the root system of G (with respect to the maximal torus associated to $\mathcal{T}$), R^+ the set of positive roots defined by the choice of C, and E the set of all positive roots which are orthogonal to Γ. Let W be the Weyl group and W_E the subgroup which stabilizes E.

The symplectic form ω on X gives us a measure

$$\frac{\omega^n}{n!}$$

called the *Liouville measure* (n is the complex dimension of X). Let us determine the push-forward of this measure by the moment map, by computing its Fourier transform:

$$A \in K \to \int_X exp(\mu(x), A)\frac{\omega_x^n}{n!}$$

By K-equivariance of μ we may assume that $A \in T$.

Theorem. *Let $X \subset \mathbf{P}(V)$ be a spherical variety. Then for every $A \in T$ we have*

$$\int_X exp(\mu(x), A)\frac{\omega_x^n}{n!}$$
$$= \prod_{\alpha\in R+} \alpha(A)^{-1}. \sum_{w\in W/W_E} det(w) \prod_{\alpha\in E}(w\alpha)(A) \int_\Gamma exp(w\gamma, A)d\gamma$$

where $d\gamma$ is the Lebesgue measure on T^*, normalized so that the torus $T^*/T^*_{\mathbf{Z}}$ has volume one.

REMARK: This result can be generalized to G–varieties which are not necessarily spherical. The above formula holds providing that the Lebesgue measure $d\gamma$ is replaced by $M(\gamma)d\gamma$, where the function M is defined as follows : For every $\gamma \in \Gamma$, the isotropy group K_γ acts on the fiber $\mu^{-1}(\gamma)$ and (for a general γ) the quotient $\mu^{-1}(\gamma)/K_\gamma$ has a canonical symplectic structure ([**DH1**] §2). Define $M(\gamma)$ to be its volume, with respect to its Liouville measure. It can be shown that the function M is locally polynomial on Γ (for a spherical variety we know from 3.2 that $\mu^{-1}(\gamma)$ is a single K_γ–orbit so that M is constant, equal to one). Full details and proofs will appear elsewhere. The case when G is a torus has been studied by Duistermaat and Heckman ([**DH1**], [**DH2**]).

Note that for a toric variety the above formula becomes very simple. In fact we have the following

Corollary. *Let $X \subset \mathbf{P}(V)$ be a closed G-variety, where G is a torus acting linearly on V. Let Γ be the image of the moment map. If G has a dense orbit in X, then the push-forward of the Liouville measure on X is the Lebesgue measure on Γ.*

PROOF: By the theorem we have

$$\int_X exp(\mu(x), A)\frac{\omega_x^n}{n!} = \int_\Gamma exp(\gamma, A)d\gamma$$

This fact has been noticed by Atiyah [**Ati**].

3.4. The degree of a spherical variety in a projective space.

Recall that the *degree* d of a closed n-dimensional subvariety X of $\mathbf{P}(V)$ is the number of points of intersection of X and a general subspace of codimension n (such a space is transversal to X). As the symplectic form ω and a general hyperplane section of X have the same cohomology class, we have $d = \int_X \omega^n$. Notation being as before, we can state the following

Theorem ([**B4**] 4.1). *The degree of X is equal to*

$$n! \int_\Gamma \prod_{\alpha\in R+-E} \frac{(\gamma,\alpha)}{(\rho,\alpha)} d\gamma$$

where ρ is half the sum of the positive roots.

PROOF: Replace A by tA in Theorem 3.3 and let $t \to 0$. The left-hand side has a limit, which is the volume of X, i.e. $\frac{d}{n!}$. Expand the right-hand side in a series of powers of t. The constant term is

$$\prod_{\alpha\in R+} \alpha(A)^{-1}. \sum_{w\in W/W_E} \prod_{\alpha\in A}(w\alpha)(A) \int_\Gamma \frac{(w\gamma, A)^{N-N_E}}{(N-N_E)!} d\gamma$$

where N (resp.N_E) is the cardinality of R^+ (resp. E). Now use the following identity on root systems ([**B4**] Lemme 4.2.1):

$$\sum_{w\in W/W_E} det(w).(\prod_{\alpha\in E} w\alpha).(w\gamma)^{N-N_E}$$
$$= (N-N_E)! \prod_{\alpha\in R+} \alpha. \prod_{\alpha\in R+-E} \frac{(\gamma,\alpha)}{(\rho,\alpha)}.$$

3.5. Common zeroes of Laurent polynomials.

As a corollary of all this theory, let us prove a result of A.Kushnirenko [**Kus**]. He considers systems of equations $\sum a_m x^m = 0$ where $m = (m_1, \ldots, m_n)$ is a multi-index of (positive or negative) integers, and x^m is the monomial $x_1^{m_1} \cdots x_n^{m_n}$. We look for their solutions in $(\mathbf{C}^*)^n$.

Theorem ([**Kus**] Théorème III'). *Consider a system S: $f_1 = \cdots f_n = 0$ where the f_i's are Laurent polynomials. Let Γ be the convex hull in $\mathbf{C}^n$ of all the m's such that x^m appears in some f_i. If the f_i's are sufficiently* *general, then the number of solutions of S in $(\mathbf{C}^*)^n$ is equal to n! times the volume of Γ.*

SKETCH OF PROOF (see also [Ati] §3 and §5): Let $m_1, \cdots m_p$ be the integral points in Γ. Define a vector space V with basis $\{e_1, \dots, e_p\}$, and let $T = (\mathbf{C}^*)^n$ act on V by: $x.e_i = x^{m_i} e_i$. Let X denote the closure in $\mathbf{P}(V)$ of the T-orbit of $x_0 = [e_1 + \cdots + e_p]$. As each f_i only involves monomials x^{m_j}, one can interpret each equation $f_i(x) = 0$ as a hyperplane section of $T.x_0$. If the f_i's are sufficiently general, they are transversal to the boundary $X - T.x_0$, and so they cut $T.x_0$ in d points, where d is the degree of X in $\mathbf{P}(V)$. It is easily seen that the image $\mu(X)$ is equal to Γ. Moreover, Theorem 3.4 applied to X gives: $d = n! \int_\Gamma d\gamma$.

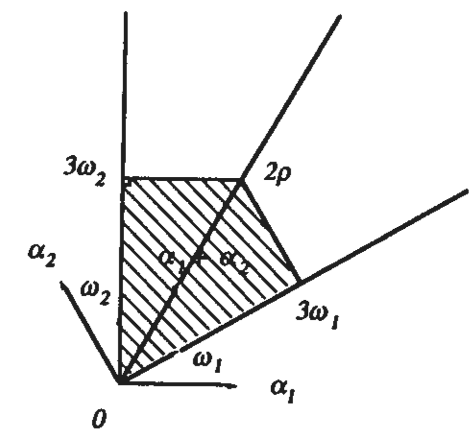
3.6. Some problems in enumerative geometry.

In the preceding example we have in fact solved the following problem: Given n sufficiently general hypersurfaces on an algebraic torus $(\mathbf{C}^*)^n$, compute the cardinality of their intersection. This kind of problem was raised by many geometers in the nineteenth century, but instead of a torus they considered spaces which arise naturally in classical projective geometry: grassmanians, flag varieties, quadrics, cubic curves ... It turns out that these spaces are very often homogeneous, and that some of them are spherical. In the latter case, we can solve the problem of "intersecting n divisors" as follows [DP2], [B4].

Let $D_1, \dots, D_n$ be hypersurfaces in an n-dimensional spherical homogeneous space G/H. From [DP2] we know that there exists a smooth compactification X of G/H such that the closures $\overline{D_1}, \dots, \overline{D_n}$ contain no G-orbit of X. As X has only finitely G-orbits, a simple generalization of a transversality theorem of Kleiman ([DP1] 6.1) shows that, for generic $g_i \in G$, the translates $g_1\overline{D_1}, \dots, g_n\overline{D_n}$ are transversal (moving the $\overline{D_i}$'s by $g_i \in G$ means putting them in general position). Moreover, the cardinality of the intersection of the $g_i D_i$'s is the product $[\overline{D_1}] \cdots [\overline{D_n}]$ of their cohomology classes in $H^*(X, \mathbf{Z})$.

So we are reduced to describing $H^2(X, \mathbf{Z})$ (which has been done in 2.3) and finding the product of n arbitrary elements of this group. It is enough to compute the powers $c_1(L)^n$, where $c_1(L)$ is the Chern class of a *positive* line bundle L (because we can find positive $L_1, \dots, L_r$ such that the $c_i(L)$'s generate the vector space $H^2(X, \mathbf{Q})$; then for $x_i > 0$ the line bundle $L = L_1^{x_1} \cdots L_r^{x_r}$ is positive; by expanding $c_1(L)^n$ we find every monomial in the L_i's). But such an L defines an embedding of X into a projective space $\mathbf{P}(V)$, and $c_1(L)^n$ is just the degree of X in $\mathbf{P}(V)$, which is given by the integral formula 3.4.

We sketch how this process works for a (very standard) example: Compute the number N of conics which are tangent to 5 given conics in general position [Kle]. Denote by D the hypersurface of $G/H = PGL(3)/PO(3)$ consisting of all conics which are tangent to a given conic. The closure of D in the variety X of complete conics (see 2.4) does not contain the closed G-orbit Y. The class of $\overline{D}$ in $H^2(X, \mathbf{Z})$ is equal to $2(\mu + \nu)$ where μ, ν are defined in 2.4. So $N = 2^5(\mu+\nu)^5$. Now X is a subvariety of $\mathbf{P}(V) \times \mathbf{P}(V^*) \subset \mathbf{P}(V \otimes V^*)$, and μ (resp. ν) is the pullback of the hyperplane section of $\mathbf{P}(V)$ (resp. $\mathbf{P}(V^*)$). So $\mu + \nu$ is the hyperplane section of X in $\mathbf{P}(V \otimes V^*)$, and $N = 32d$ where d is the degree of X in $\mathbf{P}(V \otimes V^*)$. Consider the following sketch of $\Gamma \subset C \subset T^*$ (see 3.1 to 3.3).



The arrows are the positive roots; the shaded region is Γ.

A direct computation gives

$$(\mu+\nu)^5 = 5! \int_\Gamma \frac{(\alpha_1,\gamma)(\alpha_2,\gamma)(\alpha_1+\alpha_2,\gamma)}{(\alpha_1,\rho)(\alpha_2,\rho)(\alpha_1+\alpha_2,\rho)} d\gamma = 102$$

and $N = 3264$.

3.7. Cohomology of complete conics.

We continue to use the notation of 2.4. Using the above machinery we now describe the cohomology algebra (over $\mathbf{Q}$) of the space X of complete conics.

Theorem. *The algebra $H^*(X, \mathbf{Q})$ is generated by μ and ν; the ideal of relations is generated by $2\mu^3 - 3\mu^2\nu + 3\mu\nu^2 - 2\nu^3$ and $4\mu^4 - 3\mu^2\nu^2 + 4\nu^4$.*

PROOF: Let us compute all monomials of degree 5 in μ and ν (they are called *characteristic numbers* of complete conics; see [Kle]). Let m, n be positive integers. Then $(m\mu + n\nu)^5$ is the degree of X in the projective space $\mathbf{P}(V^{\otimes m} \otimes (V^*)^{\otimes n})$. The associated polygon Γ has 0, $(2m+n)\omega_1$, $(m+2n)\omega_2$, $2(m\omega_1 + n\omega_2)$ as vertices, where ω_1, ω_2 are the fundamental weights of $PGL(3)$ ([B4] 2.7). We find that $(m\mu + n\nu)^5 = m^5 + 10m^4n + 40m^3n^2 + 40m^2n^3 + 10mn^4 + n^5$, i.e. $\mu^5 = \nu^5 = 1$, $\mu^4\nu = \mu\nu^4 = 2$, $\mu^3\nu^2 = \mu^2\nu^3 = 4$. These equalities can also be derived directly ([Kle] p.12).

Now let us prove that $\{\mu^2, \mu\nu, \nu^2\}$ is a basis of $H^4(X, \mathbf{Q})$. If $x\mu^2 + y\mu\nu + z\nu^2 = 0$ is a linear relation, then multiplying successively by μ^3, $\mu^2\nu$, $\mu\nu^2$, ν^3 we obtain $x+2y+4z = 0$, $2x+4y+4z = 0$, $4x+4y+2z = 0$, $4x + 2y + z = 0$ which implies that $x = y = z = 0$.

We can prove in the same way that $\{\mu^3, \mu^2\nu, \mu\nu^2, \nu^3\}$ is a basis of $H^6(X, \mathbf{Q})$, and that $\{\mu^4, \nu^4\}$ is a basis of $H^8(X, \mathbf{Q})$. So μ, ν generate $H^*(X, \mathbf{Q})$. It is also easily verified that $2\mu^3 - 3\mu^2\nu + 3\mu\nu^2 - 2\nu^3$ has a zero product with μ^2, $\mu\nu$ and ν^2. By Poincaré duality it must be zero. Similarly $4\mu^4 - 3\mu^2\nu^2 + 4\nu^4 = 0$.

Consider now the graded commutative algebra A over $\mathbf{Q}$ generated by x and y of degree 2, with relations $2x^3 - 3x^2y + 3xy^2 - y^3$ and $4x^4 - 3x^2y^2 + 4y^4$. As the two relations have no common divisor, A is a complete intersection, so that its Poincaré polynomial is

$$\frac{(1-z^6)(1-z^8)}{(1-z^2)^2} = (1 + z^2 + z^4)(1 + z^2 + z^4 + z^6)$$
$$= 1 + 2z^2 + 3z^4 + 3z^6 + 2z^8 + z^{10}$$

Obviously we have a surjective morphism from A to $H^*(X, \mathbf{Q})$. Since both sides have the same Poincaré polynomial, they are isomorphic.

Let me conclude with two *problems* .

(i) *How does one describe the cohomology algebra of a (smooth projective) spherical variety ?*

(ii) *How can one generalize our methods to determine the second cohomology group and characteristic numbers of a non-spherical variety ?*

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HOMOLOGY PLANES
AN ANNOUNCEMENT AND SURVEY

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We review recent advances dealing with homology planes, i.e., *non singular affine acyclic surfaces* over the complex numbers C. Here acyclic means vanishing integral reduced homology. From a topological point of view these are the simplest affine surfaces.

Before mentioning our results, we give some of the history of this topic. The first example of a homology plane different from the standard complex plane $\mathbf{C}^2$ is due to Ramanujam [R]. He exhibited a homology plane R which is even contractible. This produced a counterexample to the existing conjecture that such a surface was $\mathbf{C}^2$. He went on to characterize $\mathbf{C}^2$ as the only homology plane which is simply connected at infinity. Ramanujam's paper appeared in 1972.

It wasn't until 1987 that other homology planes were discovered. In that year Gurjar-Miyanishi [GM] produced an infinite number of examples. All but one of these was of (logarithmic) Kodaira dimension one. One was a new homology plane of Kodaira dimension 2. (The Ramanujam example has Kodaira dimension 2 as well.) Their paper in fact produced all homology planes of Kodaira dimension 1, but classification remains open because there are redundancies in their list of these surfaces. In [tDP] the authors use the work of [GM] to classify the contractible homology planes of Kodaira dimension 1 and to show that a large class of these actually occur as hypersurfaces in $\mathbf{C}^3$.

Theorem [tDP]. *Infinitely many non-isomorphic homology planes of Kodaira dimension 1 occur as hypersurfaces in* $\mathbf{C}^3$.

(This is made more precise in loc.cit.)

It wasn't previously known that homology planes could occur as hypersurfaces, but it would be surprising to the authors if all homology planes were hypersurfaces.

In their paper [GM], Gurjar-Miyanishi ask whether there are an infinite number of homology planes of Kodaira dimension 2. We announced